Associating Finite Groups with Dessins d'Enfants

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Abstract

Each finite, connected planar graph has an automorphism group G; such permutations can be extended to automorphisms of the Riemann sphere $S^2(\mathbb{R}) \simeq \mathbb{P}^1(\mathbb{C})$. In 1984, Alexander Grothendieck, inspired by a result of Gennadiĭ Belyĭ from 1979, constructed a finite, connected planar graph Δ_β via certain rational functions $\beta(z) = p(z)/q(z)$ by looking at the inverse image of the interval from 0 to 1. The automorphisms of such a graph can be identified with the Galois group $\mathrm{Aut}(\beta)$ of the associated rational function $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. In this project, we investigate how restrictive Grothendieck's concept of a Dessin d'Enfant is in generating all automorphisms of planar graphs. We discuss the rigid rotations of the Platonic solids (the tetrahedron, cube, octahedron, icosahedron, and dodecahedron), the Archimedean solids, the Catalan solids, and the Johnson solids via explicit Belyĭ maps.

Graphs

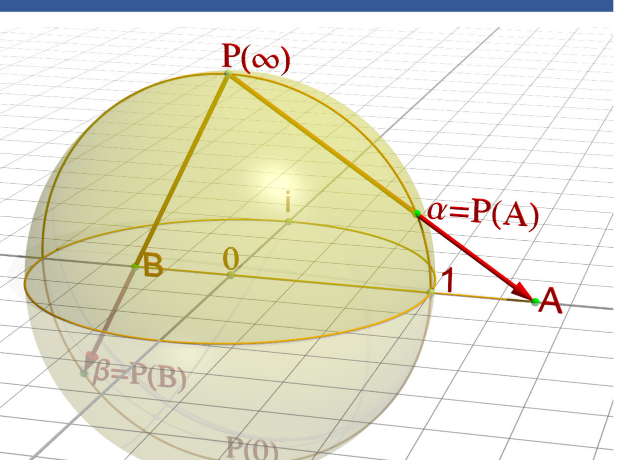
- A **(finite) graph** is an ordered pair (V, E) consisting of **vertices** V and **edges** E.

$$v = |V| = \text{the number of vertices}$$
$$e = |E| = \text{the number of edges}$$
$$f = |F| = \text{the number of faces}$$

- A **connected graph** is a graph where, given any pair of vertices z_1 and z_2, one can traverse a path of edges from one to the other.
- Two vertices z_1 and z_2 are **adjacent** if there is an edge connecting them. A **bipartite graph** is a graph where the vertices V can be partitioned into two disjoint sets B and W such that no two edges $z_1, z_2 \in B$ (respectively, $z_1, z_2 \in W$) are adjacent.
- A **planar graph** is a graph that can be drawn such that the edges only intersect at the vertices.

The Sphere as The Extended Complex Plane

Through stereographic projection, we can establish a bijection between the unit sphere $S^2(\mathbb{R})$ and the extended complex plane $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.



Define **stereographic projection** as that map from the unit sphere to the complex plane.

$$S^2(\mathbb{R}) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$$
$$(u, v, w) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right) \mapsto x + i\,y = \frac{u + i\,v}{1 - w}$$

Belyĭ maps

Let $\beta : X \to \mathbb{P}^1(\mathbb{C})$ be a meromorphic function on a Riemann surface X. We say $z \in X$ is a **critical point** if $\beta'(z) = 0$, and $w \in \mathbb{P}^1(\mathbb{C})$ is a **critical value** if $w = \beta(z)$ for some critical point $z \in X$. A rational function $\beta : X \to \mathbb{P}^1(\mathbb{C})$ which has at most three critical values $\{0, 1, \infty\}$ is called a **Belyĭ map**.

Dessin d'Enfant

Fix a Belyĭ map $\beta : X \to \mathbb{P}^1(\mathbb{C})$. Denote the preimages

$$B = \beta^{-1}(\{0\}) \qquad W = \beta^{-1}(\{1\}) \qquad E = \beta^{-1}([0,1]) \qquad X \xrightarrow{\beta} \mathbb{P}^1(\mathbb{C})$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\left\{\begin{matrix}\text{black}\\ \text{vertices}\end{matrix}\right\} \qquad \left\{\begin{matrix}\text{white}\\ \text{vertices}\end{matrix}\right\} \qquad \left\{\text{edges}\right\} \qquad \mathbb{R}^3$$

The bipartite graph $\Delta_\beta = (V, E)$ with vertices $V = B \cup W$ and edges E is called **Dessin d'Enfant**. We embed the graph on X in 3-dimensions.

Platonic Solids

Rigid Rotations of the Platonic Solids

We have an action $\circ : PSL_2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$.

- $Z_n = \langle r \mid r^n = 1 \rangle$ and $D_n = \langle r, s \mid s^2 = r^n = (s\,r)^2 = 1 \rangle$ are the rigid rotations of the **regular convex polygons**, with

$$r(z) = \zeta_n\, z \qquad \text{and} \qquad s(z) = \frac{1}{z}.$$

- $A_4 = \langle r, s \mid s^2 = r^3 = (s\,r)^3 = 1 \rangle$ are the rigid rotations of the **tetrahedron**, with

$$r(z) = \zeta_3\, z \qquad \text{and} \qquad s(z) = \frac{1 - z}{2\,z + 1}.$$

- $S_4 = \langle r, s \mid s^2 = r^3 = (s\,r)^4 = 1 \rangle$ are the rigid rotations of the **octahedron** and the **cube**, with

$$r(z) = \frac{\zeta_4 + z}{\zeta_4 - z} \qquad \text{and} \qquad s(z) = \frac{1 - z}{1 + z}.$$

- $A_5 = \langle r, s \mid s^2 = r^3 = (s\,r)^5 = 1 \rangle$ are the rigid rotations of the **icosahedron** and the **dodecahedron**, with

$$r(z) = \frac{(\zeta_5 + \zeta_5{}^4)\,\zeta_5 - z}{(\zeta_5 + \zeta_5{}^4)\,z + \zeta_5} \qquad \text{and} \qquad s(z) = \frac{(\zeta_5 + \zeta_5{}^4) - z}{(\zeta_5 + \zeta_5{}^4)\,z + 1}.$$

Platonic, Archimedean, Catalan, and Johnson Solids

- A **Platonic solid** is a regular, convex polyhedron. They are named after Plato (424 BC – 348 BC). Aside from the regular polygons, there are five such solids.
- An **Archimedean solid** is a convex polyhedron that has a similar arrangement of nonintersecting regular convex polygons of two or more different types arranged in the same way about each vertex with all sides the same length. Discovered by Johannes Kepler (1571 – 1630) in 1620, they are named after Archimedes (287 BC – 212 BC). Aside from the prisms and antiprisms, there are thirteen such solids.
- A **Catalan solid** is a dual polyhedron to an Archimedean solid. They are named after Eugène Catalan (1814 – 1894) who discovered them in 1865. Aside from the bipyramids and trapezohedra, there are thirteen such solids.
- A **Johnson solid** is a convex polyhedron with regular polygons as faces but which is not Platonic or Archimedean. They are named after Norman Johnson (1930) who discovered them in 1966. There are ninety-two Johnson solids.

Proposition (Wushi Goldring, 2012)

Let $\varrho(w)$ be a rational function. The composition $\varrho \circ \beta$ is also a Belyĭ map for every Belyĭ map $\beta : X \to \mathbb{P}^1(\mathbb{C})$ if and only if ϱ is a Belyĭ map which sends the set $\{0, 1, \infty\}$ to itself.

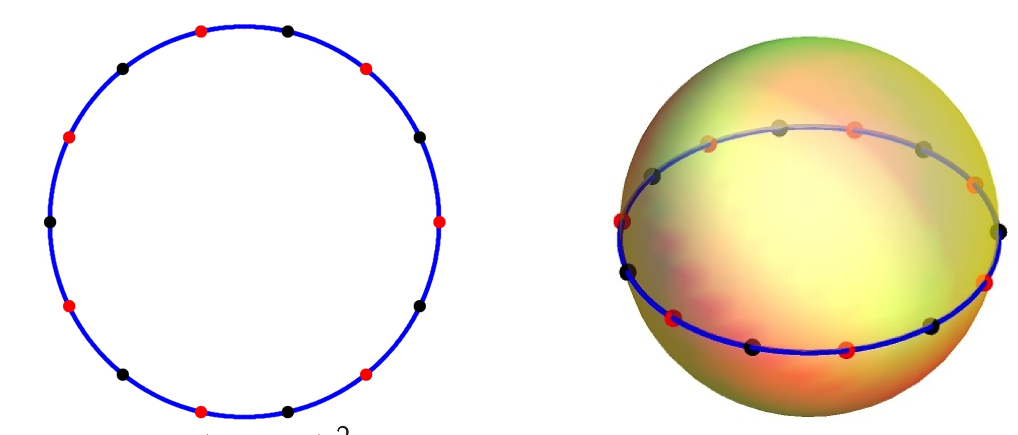
Proposition (Nicolas Magot and Alexander Zvonkin, 2000)

The following ϱ are Belyĭ maps which send the set $\{0, 1, \infty\}$ to itself.

$$\varrho(w) = \begin{cases} -(w-1)^2/(4\,w) & \text{is a } \textbf{rectification}, \\ (4\,w-1)^3/(27\,w) & \text{is a } \textbf{truncation}, \\ 1/w & \text{is a } \textbf{birectification}, \\ (4-w)^3/(27\,w^2) & \text{is a } \textbf{bitruncation}, \\ 4\,(w^2-w+1)^3/(27\,w^2\,(w-1)^2) & \text{is a } \textbf{rhombitruncation}, \\ (w+1)^4/(16\,w\,(w-1)^2) & \text{is a } \textbf{rhombification}, \\ \dfrac{7496192\,(w+\theta)^5}{25\,(3+8\,\theta)\,w\,(88\,w-(57\,\theta+64))^3} & \text{is a } \textbf{snubification}, \end{cases}$$

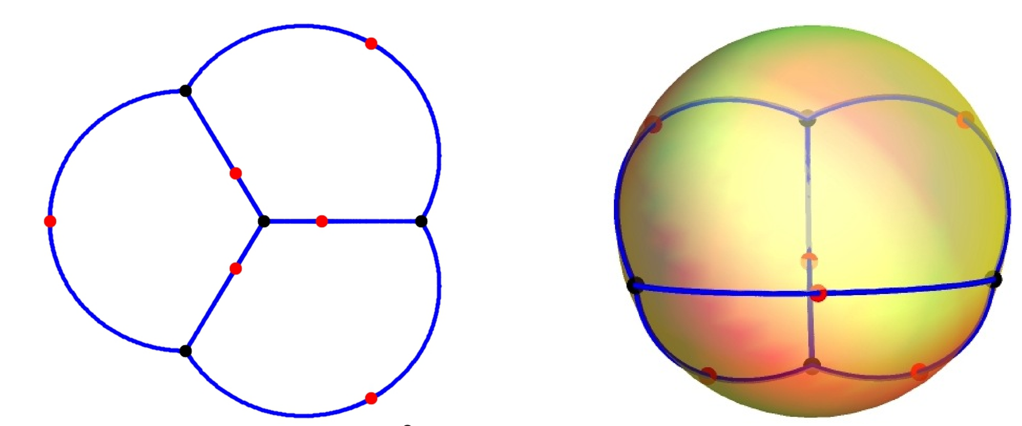
where $\theta^2 - (7/64)\,\theta + 1 = 0$.

Rotation Group D_n: Regular Convex Polygon



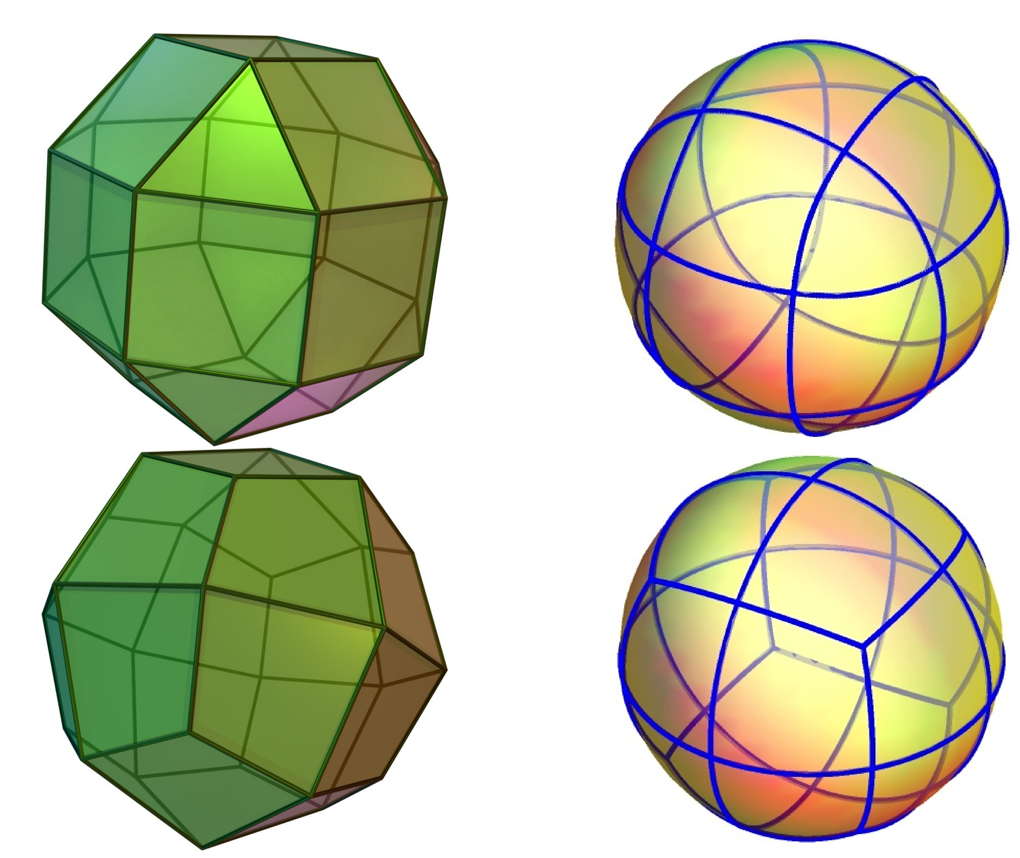
$$\beta(z) = \frac{(z^n + 1)^2}{4\,z^n}: \qquad v = n + n, \quad e = 2 \cdot n, \quad f = 2$$

Rotation Group A_4: Tetrahedron

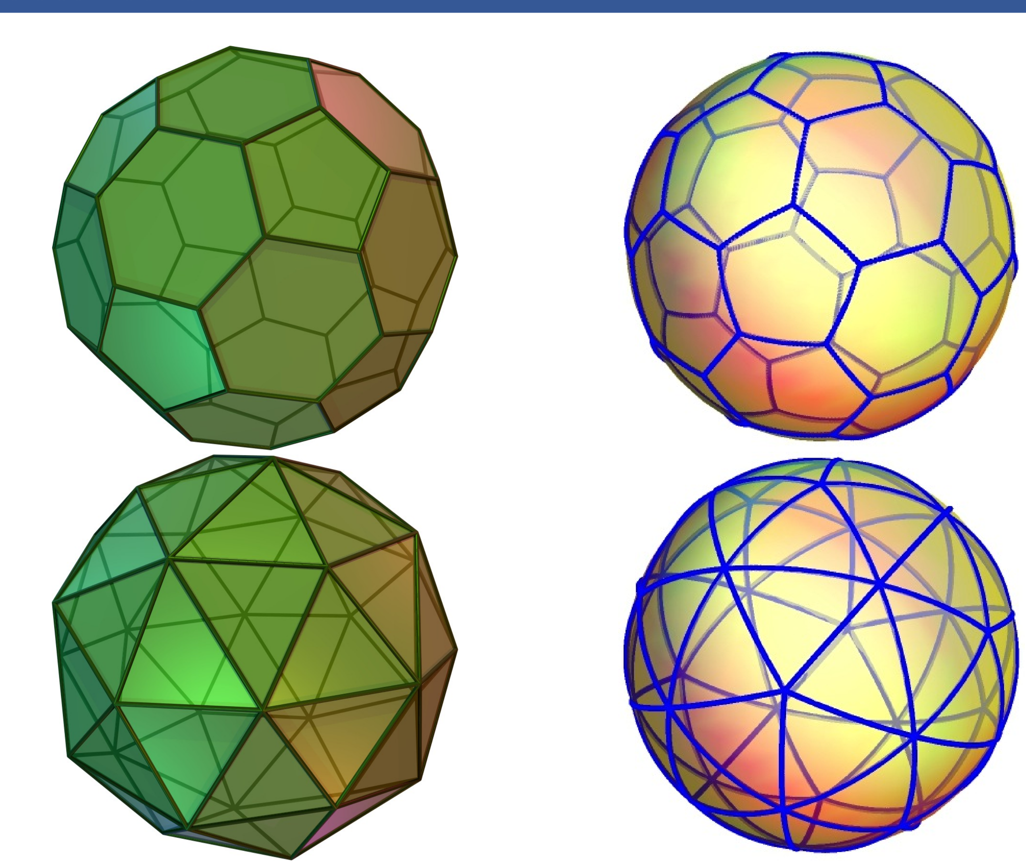


$$\beta(z) = -\frac{64\,z^3\,(z^3 - 1)^3}{(8\,z^3 + 1)^3}: \qquad v = 4 + 6, \quad e = 2 \cdot 6, \quad f = 4$$

Rotation Group S_4: Rhombicuboctahedron / Deltoidal Icositetrahedron



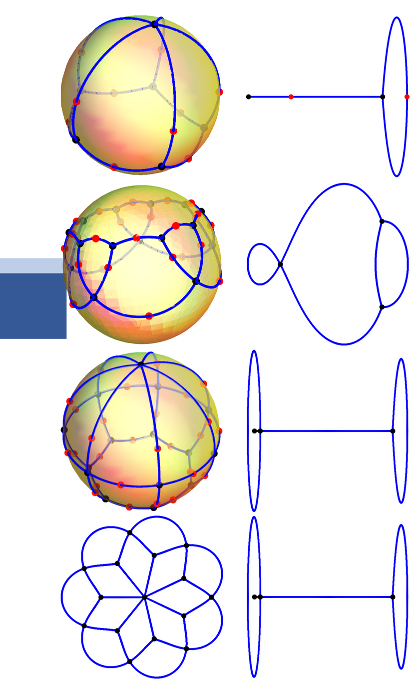
Rotation Group A_5: Truncated Icosahedron / Pentakis Dodecahedron



Approach

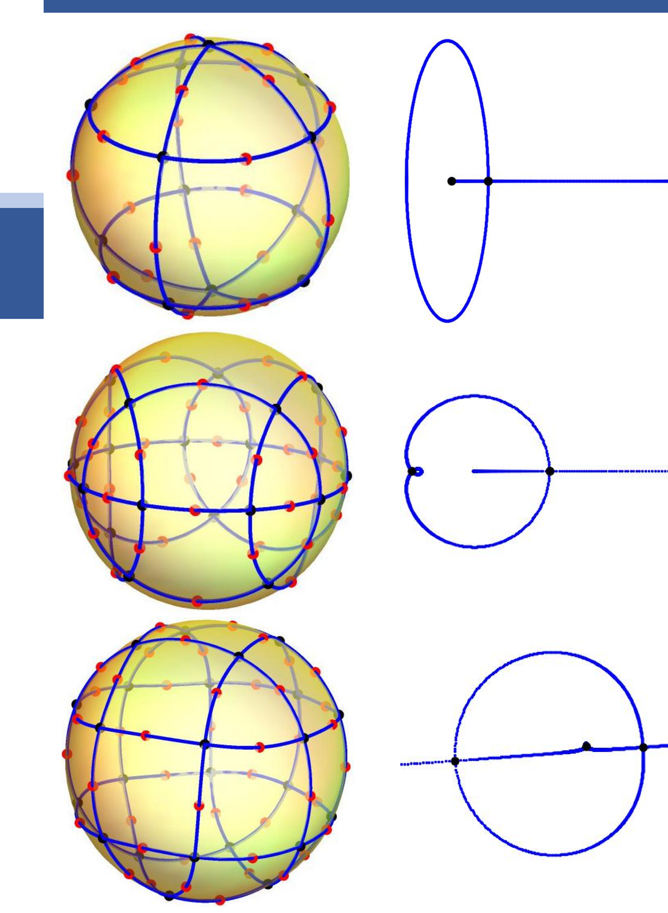
Following Magot and Zvonkin, reduce to easier cases using "hypermaps" $\phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, then composing $\beta = \phi \circ f$ where $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a Belyĭ map as a function of either z^n or $4\,z^n/(z^n+1)^2$ such that $\mathrm{Aut}(f) \simeq Z_n$ or $\mathrm{Aut}(f) \simeq D_n$, respectively.

Hypermaps: Rotation Group Z_n



- Wheel/Pyramids (J_1, J_2)
- $\phi(w) = \dfrac{w^3\,(w+8)}{64\,(w-1)}$
- Cupola (J_3, J_4, J_5)
- $\phi(w) = \dfrac{4w^4(w^2 - 20w + 105)^3}{(7w-48)^3(3w-32)^2(5w+12)}$
- Elongated Pyramids (J_7, J_8, J_9)
- $\phi(w) = \dfrac{4\,(835+872\sqrt{2})\,w^4\,(w-1)^3\,\left[(11+8\sqrt{2})\,w+1\right]}{\left[(8+9\sqrt{2})\,w+1\right]^3\left[(8-5\sqrt{2})\,w-1\right]}$
- Diminished Trapezohedron
- $\phi(w) = \dfrac{4\,(835-872\sqrt{2})\,w^4\,(w-1)^3\,\left[(11-8\sqrt{2})\,w+1\right]}{\left[(8-9\sqrt{2})\,w+1\right]^3\left[(8+5\sqrt{2})\,w-1\right]}$

Hypermaps: Rotation Group D_n



- Elongated Bipyramid (J_{14}, J_{15}, J_{16})
- $\phi(w) = \dfrac{w\,(32\,w-5)^4}{(80\,w+1)^3}$
- Orthobicupola (J_{27}, J_{28}, J_{30})
- $\phi(w) = \dfrac{(w-2.411)^4\,(w+0.138)^4}{w\,(w+3.086)^3\,(w-0.441)^4}$
- Elongated Gyrobicupola (J_{36}, J_{37}, J_{39})
- $\phi(w) = \dfrac{\left[(w^3+(0.739+0.223\,i)w^2-(0.754+0.034\,i)\,w+(0.002-0.020\,i)\right]^4}{w\left[w-(0.041-0.283\,i)\right]^3\left[w^2-(2.004+0.189\,i)\,w+(0.005+0.324\,i)\right]^4}$

Theorem (E. Baeza, L. Baeza, C. Lawrence, and C. Wang, 2014)

There are explicit Belyĭ maps β for

- Wheel/Pyramids (J_1, J_2)
- Cupola (J_3, J_4, J_5)
- Elongated Pyramids (J_7, J_8, J_9)
- Diminished Trapezohedron

which have rotation group $\mathrm{Aut}(\beta) \simeq Z_n$; and

- Bipyramid (J_{12}, J_{13})
- Elongated Bipyramid (J_{14}, J_{15}, J_{16})
- Gyroelongated Bipyramid (J_{17})
- Orthobicupola (J_{27}, J_{28}, J_{30})
- Gyrobicupola (J_{29}, J_{31})
- Elongated Orthobicupola (J_{35}, J_{38})
- Elongated Gyrobicupola (J_{36}, J_{37}, J_{39})
- Dipole/Hosohedron
- Truncated Trapezohedron
- Bifrustum/Truncated Bipyramid

which have rotation group $\mathrm{Aut}(\beta) \simeq D_n$.

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